

Paul Oslington

# Statistics maths primer

for economics,  
commerce  
and business  
administration  
students



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# Why this book was written

This book grew out of seeing students struggle with the increasing mathematics requirements of university economics, commerce and business courses. For many otherwise capable students, a lack of skills and confidence with mathematics has been a significant hindrance to their progress in those courses — either because of inadequate earlier teaching in mathematics, because their mathematics has grown rusty since their previous studies, or because they are unable to apply the mathematics they know to economics. Sadly, many students have come to fear mathematics and believe they are ‘no good at maths’.

Thus, the book was written to help you improve or refresh your maths, so that you can cope with the mathematical requirements of undergraduate and some postgraduate programs in economics, commerce and business administration. Only a familiarity with numbers and basic arithmetic is assumed — such as from a dimly remembered junior school mathematics course. It aims to help you develop the skills and confidence to succeed in university courses.

# How to use this book

This mathematics primer is designed for use as a self-study guide for students or groups of students taking undergraduate economics, commerce or business degrees. As a student you will need the material in this book for many courses in your degree, and later in the workplace. Use it in conjunction with your textbooks for those courses. It is a shorter, easier book than the existing books for semester-length university courses in mathematics for economists or mathematical economics. Some of these more advanced books are included in the suggestions for further reading. Teachers and students of shorter orientation courses in mathematics may find it useful as a textbook.

The book covers all the mathematics you are likely to encounter in core courses in an undergraduate economics, commerce or business degree. The basic material in chapters 1 to 5 needs to be mastered by all students. Without this mastery you will be unable to do assignment and examination questions in your courses. The remaining part, chapters 6 to 9, contains material that some students will need and others will not. In these later chapters some advanced topics are discussed, for instance in chapter 6 constrained optimisation, but for these advanced topics the discussion is aimed at giving you a basic understanding of the concept rather than helping you to carry out the calculations and master the topic. If you want to master these advanced topics, look up one of the suggestions for further reading.

Now you have bought the book, I want you to use it well. Buying it will please the publisher and bookshop, but leaving

it on your shelf, or even putting it under your pillow at night, will not improve your mathematical skills. I suggest you look briefly through the whole book so that you know what is in it, preferably before your courses commence, or in the first few weeks of semester. I have written the book so that it can be actually read in a way that many mathematics books cannot — equations are incorporated in the body of the text and the illustrative examples in the text have been kept as simple as possible. Then begin working on areas of your maths that are weak and which you know you will need (perhaps from the course outlines, perhaps from your teacher — I hope not from having failed the course last year!). In addition to this early work, there will be times when there is maths in lectures or assignments that you do not understand. When this happens try to find the relevant part of the book and revise those topics. Do not put it off — it is now fresh in your mind, and spare time tends to get less abundant rather than more so as the semester goes on. The other piece of advice I would give you, and I cannot emphasise this too strongly, is not just to read through the text, but to attempt the exercises. When attempting the exercises, take the trouble to set out your ideas and working clearly — it will help you to minimise errors and facilitate revision later. It will also get you into the habit of setting out your working, which is valuable for examinations where you will be awarded marks for working which demonstrates some understanding of the problem, even if you do not get the correct answer. Maths, like any skill, is learnt by doing rather than watching. Working through the exercises will help your confidence — yes, you really *can* do maths!

Working through this book will be frustrating at times. Do not give up too easily — mathematical problems will often suddenly make sense after you have struggled with them for a while, but you cannot get to that point without spending some time on the problem. Moreover, it is satisfying to have worked something out. I believe everyone can do maths, at least to the level of competence you will need to complete your studies.

Finally, a few words about the place of mathematics in economics and commerce courses. Over recent decades economics

and commerce courses have become increasingly mathematical. This is contentious, as those without a strong mathematical background, often women and members of minority groups, are then excluded from the discussion. While it can also be argued that mathematical economics ignores important ideas which cannot be expressed in mathematical terms, you will also find that mathematics is a precise and powerful way of expressing ideas. Using mathematics has led to many new insights in economics, commerce and business. Mathematics also has made possible the quantification and testing of economic theories in a way that would not have been possible if these theories were expressed only in words. For the advocates of mathematics in economics, mathematics is just a useful alternative language — there is nothing mysterious or contentious about it. It may be just a language like any other, but then like all language it has a social context and effects on the practice of economics within society.

As a beginning student, whatever views you take of the mathematisation of economics and commerce will not alter the immediate reality of needing to acquire these skills to progress in your studies, but my hope is that as a mathematically competent and confident graduate you will be valuable to future employers and be able to participate in these and other important debates.



CHAPTER 1

# NUMBERS

This chapter covers some basic mathematical rules you will need to understand later chapters. My advice would be to look through the chapter, and if any of it is unfamiliar go back and spend more time ensuring you understand it before moving on to the later chapters.

## The World of Numbers

In this book you will be dealing with all kinds of numbers, and it will be good to know something about them before you meet them. They are best introduced using the number line in figure 1.1.

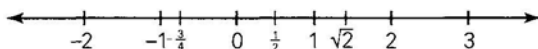


Figure 1.1 Number line

Some numbers you already know are the **counting numbers** — 1, 2, 3 and so on. If we consider zero and the negatives of the counting numbers, we have the **integers**. There are also the numbers in between which can be written as one integer divided by another, like  $\frac{1}{2}$  or  $-\frac{3}{4}$ , and considering these as well as the integers gives us the **rational numbers**. The integer that is being divided (the top of the fraction) is called the **numerator**, and the integer it is divided by (the bottom of the fraction) is called the **denominator**. This is not all there is in the world of numbers; there are also some weird numbers, like the square root of 2 (in other words, the number which multiplied by itself gives 2), called **irrational numbers**. They are weird because we can never write their value down exactly; the best we can do is show other numbers they must be between on the number line. The square root of 2 is between 1.41 and 1.42. The rational numbers plus these irrational numbers give us the **real numbers**, which are about as far as we will go in this book.

## Order of Operations

Some of the operations we can carry out on numbers are **addition** +, **subtraction** −, **multiplication** ×, and **division** ÷ or /.

A rule you probably know already is that a negative number multiplied or divided by a negative number gives a positive number, and that a negative multiplied or divided by a positive number gives a negative number. Think of multiplication by a negative as changing the sign. **Division by zero** is not possible — the answer is undefined.

What happens if we are asked to do several operations, for instance if we are asked to evaluate  $2 + 2 \times 3$ ? Do we work out the addition first, giving  $4 \times 3 = 12$ , or do the multiplication first, giving  $2 + 6 = 8$ ? Fortunately mathematicians have worked out a rule that any multiplications and divisions are evaluated before any additions or subtractions, so that the correct answer to our problem was 8. If there are any powers or other operations, which you will find out about later in this chapter, they are worked out before multiplications or divisions.

Often, though, we want to override these rules for the order of operations, and this is done by using **brackets**. In the above example, if we want the addition evaluated first we would put brackets around it:  $(2 + 2) \times 3$ , and the correct answer would now be 12. Brackets can be  $()$  or  $\{\}$  or  $[\ ]$ . Where there are several sets of brackets you work outwards, evaluating the inner set of brackets first and then the outer. For instance, to evaluate  $[(2 + 2) \times 3] + 1$  you first evaluate  $2 + 2$ , then multiply by 3, then add 1, giving 13.

Be careful with fractions — this is where a lot of confusion occurs. If you write  $1/2 + 3$  then the division is evaluated first, giving an answer of  $3\frac{1}{2}$ . I hope you didn't mean  $1/(2 + 3)$ , which is  $1/5$ . It is much clearer to use a horizontal line than a slash:  $\frac{1}{2} + 3$  and  $\frac{1}{2+3}$  are not as easily confused.

## Powers

Often in this book you will meet numbers like  $5^2$ , which is 5 to the power of 2, or 5 squared. It means 5 multiplied twice, which is  $5 \times 5 = 25$ . Similarly  $5^3$  is 5 to the power of 3, or 5 cubed, which is  $5 \times 5 \times 5 = 125$ .

Any number to the **power one** is itself, so that  $5^1 = 5$ . Numbers without powers are really numbers to the power one.

Any number to the **power zero** is 1, for instance  $5^0 = 1$ .

What about something like  $5^{-2}$ ? The **power is negative** indicating the reciprocal or inverse, which means 1 divided by  $5^2$ , which is  $1/5^2$  or  $1/25$ . So  $5^{-3}$  will be  $1/5^3$  or  $1/125$ . What about  $25^{1/2}$ ? This is a **power that is a fraction** and means the second root or square root of 25 (the number which multiplied by itself gives 25), which is 5.  $\sqrt{25}$  is another way of writing the square root of 25. Similarly  $125^{1/3}$  is the third root or cube root of 125, which is 5. How about  $125^{2/3}$ ? This is the cube root of 125, which was 5, which we then square to give 25.

You should know that 25 or any other positive number actually has two square roots because both 5 and  $-5$  multiplied by themselves give 25. Note that negative numbers do not have a square root, because there is no real number that multiplied by itself gives a negative number.

A problem you will sometimes face is to add or multiply numbers with powers, and there are mathematical rules which cover this. If you **are multiplying numbers to the same power**, then multiply the numbers and put them to this power. For example  $2^2 \times 3^2$  is  $(2 \times 3)^2 = 6^2$ . If you **are multiplying the same number to different powers** then add the powers. For example  $2^2 \times 2^3$  is  $2^{2+3} = 2^5$ . If you **are putting a number to a power to another power**, then multiply the powers. For example  $(2^2)^3$  is  $2^{2 \times 3} = 2^6$ .

## Some Special Symbols

The most common other symbols you will encounter in economics, commerce and business courses are:

- $\infty$  is the symbol for infinity.
- $\pi$  which is **pi**, one of the Greek letters. This is the ratio of the distance around a circle (called the circumference) to the distance across it, through the centre (called the diameter). It is an irrational number that is around about 3.142.
- $e$  is the **natural number**. In nature things often grow at rates related to  $e$ , and it appears often. It is also irrational and roughly 2.72.
- log is an abbreviation for **logarithm**. Logarithms have a base, and the most common is  $e$ , written fully as  $\log_e$ ,

but it could 10, written  $\log_{10}$ , or any other number. The logarithm of a number is the power the base must be raised to to give the number. For instance if the logarithm to base  $e$  of a number is 2, then it means the number is equal to  $e^2$ , and the number is about 7.39.

$\Sigma$  the Greek letter **sigma**, which means add up whatever follows this symbol.

! is the symbol for the **factorial** operation. It asks you to multiply the number by all the counting numbers below it. For example  $4! = 4 \times 3 \times 2 \times 1 = 24$ .

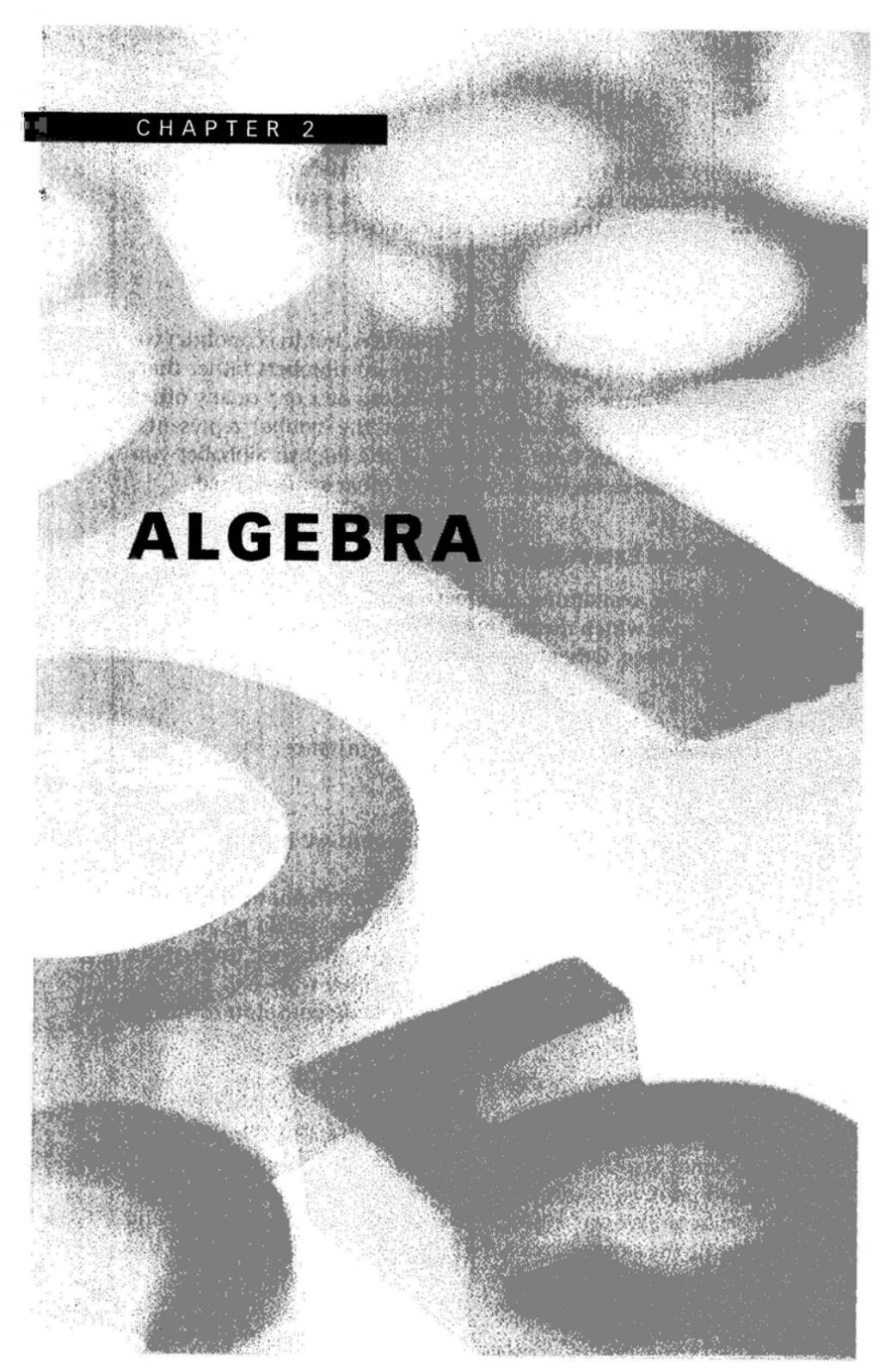
| | is the symbol for the **absolute value** operation. The absolute value of a number is its distance away from zero on the number line. For a positive number the absolute value will be just the number, so  $|5| = 5$ , and for a negative number you just delete the negative sign, so  $|-5| = 5$ .

**EXERCISES****Order of Operations**

1. Evaluate  $4(3 + 1) \times 3 + 1$
2. Evaluate  $4 \times 3^2$
3. Evaluate  $\left(\frac{2}{3}\right)^2$
4. Evaluate  $7 - (6 + 3)$
5. Evaluate  $7 - (-3)$

**Powers**

6. Evaluate  $16^{1/2}$
7. Evaluate  $8^{2/3}$
8. Evaluate  $16^{-1/2}$
9. Evaluate  $\frac{5^3}{5^2}$
10. Evaluate  $(4^{2/3})^3$
11. Evaluate  $(3^2 \cdot 2^2)^2$
12. Evaluate  $7^2 \cdot 2^2$
13. Evaluate  $\left(\frac{2}{3}\right)^2$
14. Evaluate  $7^2 + 8^2$
15.  $\sqrt{4 \times 5 + 5}$



CHAPTER 2

# ALGEBRA

Algebra is a set of skills you will need again and again as you work through this book and progress in your studies. Make sure you master this chapter before moving on.

## Symbols

The previous chapter dealt with numbers, but in economics we often work with symbols that represent numbers rather than numbers themselves. These symbols can be  $x$  or  $y$  or any other letter that helps us remember what the symbol represents. Because there are only 26 letters in the English alphabet you will sometimes see Greek letters and other symbols used.

## Terms and Expressions

Something containing numbers or symbols is a **term**, for instance  $3x$  which means multiplying the number that  $x$  represents by three. Other terms include

$$7, -x, 3x^2, \frac{x}{3}$$

**Expressions** are made up of terms, for instance

$$3x + 7 - x + 3x^2 + \frac{x}{3}$$

is an expression made up of all the terms we have mentioned so far.

Manipulating expressions is a vital mathematical skill and we will now see how it is done. The reason we do it is to write expressions more simply so that we can better understand their meaning, or work with them in other ways.

One type of manipulation is **collecting common terms**. In the expression

$$3x + 7 - x + 3x^2 + \frac{x}{3}$$

$x$  is a common term being multiplied by 3 in the first term, does not appear in the second term, is multiplied by  $-1$  in the third term, by 3 and by  $x$  in the fourth term, and by  $\frac{1}{3}$  in the fifth term. We can collect all these operations on  $x$  together and rewrite the expression as  $(3 - 1 + 3x + \frac{1}{3})x + 7$ . Since  $3 - 1 = 2$  this

can be written more simply as  $(2 + 3x + \frac{1}{3})x + 7$ . Note that the second term 7 was not a term that we collected so it is still there just the same as in the original expression. This new expression is simpler than the original expression but has exactly the same value as the original expression. If you are unsure that the value is the same, pick a number for  $x$  and check it by evaluating the original and new expressions.

A variation on collecting terms is writing an expression containing fractions with a **common denominator**. (Remember from the last chapter that the top of a fraction is called the numerator and the bottom is called the denominator.) The common denominator is usually found by multiplying the denominators of the fractions in the expression. After you have found the common denominator you then multiply the numerator and denominator of each fraction by the number needed to make the denominator of each fraction equal to this common denominator.

Consider  $\frac{x}{2} + \frac{x}{3}$

Both fractions in this expression can be written as a fraction of 6, so 6 is the common denominator. Multiplying the numerators and denominators:

$$\frac{x}{2} \text{ by } 3 \text{ gives } \frac{3x}{6}$$

and  $\frac{x}{3} \text{ by } 2 \text{ gives } \frac{2x}{6}$

so the expression equals  $\frac{3x + 2x}{6}$

which simplifies to  $\frac{5x}{6}$

This procedure can be applied to more complex expressions containing fractions.

Another important type of manipulation is **expanding an expression**. Consider  $3x(4x - 2)$ . Part of this expression is bracketed, but we may want to write it without brackets so we can see more clearly what it is. To do this, multiply each of the terms inside the brackets by the term outside the brackets and

add up the results, which gives  $12x^2 - 6x$ . Now consider the expression  $(x + 4)(x - 3)$ . This is harder because there are two sets of brackets, but we can still expand it. We multiply each term inside the first bracket by each term inside the second bracket and then collect terms. For our expression  $(x + 4)(x - 3)$  multiplying gives  $x^2 - 3x + 4x - 12$ , and then collecting terms gives  $x^2 + x - 12$ .

The reverse of expanding an expression is **factorising an expression**. Since the expansion of  $3x(4x + 2)$  was  $12x^2 + 6x$ , the factorisation of the expression  $12x^2 + 6x$  is  $3x(4x + 2)$ . Factorising involves finding something that is common to each of the terms in the expression we are trying to factorise, and then working out what this common factor is being multiplied by in each term. For instance factorising  $12x^2 + 6x$  involved recognising that  $3x$  is common to both  $12x^2$  and  $6x$ .  $3x$  is being multiplied by  $4x$  in the  $12x^2$  and by  $2$  in  $6x$ . Thus the factorisation is  $3x(4x + 2)$ .

Factorising expressions like  $x^2 + x - 12$  is a little more difficult. In general the factors of an expression  $x^2 + bx + c$ , where  $b$  and  $c$  are constants, will be  $(x + f)(x + g)$ , where  $f$  and  $g$  are constants which added give  $b$  and multiplied give  $c$ . (You can check this by expanding out  $(x + f)(x + g)$  and verifying that  $b = g + f$  and  $c = gf$ .) When you are actually factorising you will have to try different values for  $f$  and  $g$  until you find the right ones. For  $x^2 + x - 12$  the factors  $4$  and  $-3$  add up to  $1$  and multiply to  $-12$ , so the factorisation is  $(x + 4)(x - 3)$ . Factorising expressions takes some practice. Remember you can always check your factorisation answer by expanding it and making sure you get the expression you factorised.

Some rules can save you a bit of time when you are factorising expressions. The factorisation of  $x^2 + 2xc + c^2$ , where  $c$  is a constant, is  $(x + c)^2$ . For example,  $x^2 + 8x + 16$  factorises to  $(x + 4)^2$ . Another rule is that the factorisation of  $x^2 - c^2$  is  $(x + c)(x - c)$ . For example the factorisation of  $x^2 - 16$  is  $(x + 4)(x - 4)$ .

## Equations

Now that you are confident about expressions and manipulating them we can start talking about equations. An **equation** is

two expressions linked together by an = sign, where the expressions on both sides of the = sign are only equal for some or usually just one value of the symbol (which we will now often call the variable).

Let's try to **solve an equation**, in other words find this value. Consider the equation  $3x + 7 = 13$ . To find the value of  $x$  which solves it, manipulate the equation to try to get just  $x$  on its own on one side of the equation. When you are manipulating the equation be sure to carry out exactly the same operations on both sides of the equation so that the equality is preserved. Here one step towards getting  $x$  on its own would be to subtract 7 from both sides. This gives  $3x = 6$ . If we then divide both sides of the equation by 3, we have  $x = 2$ , which is the solution. A good habit to get into when solving equations is to check your answer. Put  $x = 2$  into the equation, and both sides are equal, so we are sure this is the solution.

Now consider a slightly harder problem,  $2 - 6x = 11$ . Subtract 2 from both sides to give  $-6x = 9$ . Divide by  $-6$  to give  $x = -1.5$ . Check that this satisfies the equation.

Consider another example, the equation

$$\frac{2x+3}{5} = 3$$

To solve this, multiply both sides by 5 to give  $2x + 3 = 15$ , then subtract 3 from each side giving  $2x = 12$ , so  $x = 6$  is the solution to the equation.

One thing you should recognise is that while the simple equations we have considered have one solution, many other equations have no solution or have many solutions. These equations typically have  $x$  to powers other than 1, so don't worry about it unless you see this in an equation you are trying to solve.

## Identities

Some expressions linked by an = sign are equal for all values of the variable in the equation. They are called **identities**. For instance  $x^2 + x - 12 = (x + 4)(x - 3)$  for all values of  $x$  — we know this because these are the expressions in the previous section on factorisation and the answer we got when we factorised —

and it is thus an identity. The best way to check whether we have an identity is to try the various manipulations described in the last section and see if we can write the expressions on both sides of the  $=$  sign in exactly the same way.

Identities are usually of less interest to us than equations.

## Inequalities

Just as an equation was two expressions linked by an  $=$  sign, **inequalities** are two expressions linked by an inequality sign. These inequality signs are  $>$ , which means greater than,  $<$ , which means less than,  $\geq$ , which means greater than or equal to, and  $\leq$ , which means less than or equal to.

Inequalities are solved in much the same way as equations, but remember one special rule. When multiplying or dividing both sides of an inequality by a negative number you reverse the direction of the inequality sign. Consider  $5 - x > 1$ . Subtract 5 from both sides to give  $-x > -4$ . Multiply both sides by  $-1$  and reverse the direction of the inequality to give  $x < 4$ . To check this solution, pick a value of  $x$  which is less than 4, say 1, and ensure it satisfies the inequality we were trying to solve.

## Systems of Equations

Often in economics and finance you will encounter **systems of simultaneous equations**. For example, consider the equations  $q = 10 + 2p$  and  $q = 110 - 3p$ . (These are supply and demand equations from economics which link the price  $p$  and quantity  $q$  of a commodity.) The solution to find is a value of  $p$  and a value of  $q$  which satisfy both equations. To find it, begin by substituting the value of  $q$  from one of the equations into the other equation. This gives  $10 + 2p = 110 - 3p$ . Now solve for  $p$  by manipulating, just as you would for any other equation you are trying to solve. Subtract 10 from both sides to give  $2p = 100 - 3p$ . Add  $3p$  to both sides to give  $5p = 100$ , and  $p = 20$ . To finish the problem and find the value of  $q$ , substitute  $p = 20$  into one of the equations. Substituting it into either of the equations gives  $q = 50$ .

Notice that in this system of equations we just solved there were two equations and two variables  $p$  and  $q$ . We found a solution and there was only one solution. What about if we just had the supply equation on its own? Then there would be an infinite number of solutions; for every value of  $p$  we picked there would be a value of  $q$  which would satisfy the equations. What about if we had another equation like  $q = 40 - p$  which had to be satisfied as well as the original equations? Then there would be no value of  $p$  and  $q$  which would satisfy all the equations, in other words there would be no solution. This suggests an important rule for systems of equations. When there are more variables than equations we should expect an infinite number of solutions; when there are fewer variables than equations we should expect no solution, and when there are exactly as many variables as equations there will usually, but not always be, a unique solution.

## EXERCISES

## Terms and Expressions

1. Expand  $-3(9x + 2)$
2. Expand  $(x + 3)^2$
3. Expand  $(x + 3)(9x + x - 7)$
4. Simplify  $3x^2 + 7x + 2x^2 - 3 - 2x$
5. Simplify  $3x^2 + 7x - (2x^2 - x + 6)$
6. (Difficult) Simplify  $\frac{x^2 + 2x - 3}{x^2 + 4x + 3}$
7. (Difficult) Simplify  $\frac{x^2 + x - 2}{x^2 + 8x + 12}$
8. Simplify  $\frac{x}{4} + \frac{x}{3}$
9. Simplify  $\frac{2x}{7x} - \frac{1}{3x}$
10. Factorise  $x^3 + 5x - 6$
11. Factorise  $2x^2 - 5x - 3$
12. Factorise  $x^2 - 25$
13. (Difficult) Factorise  $4x^{1/3} - x^{2/3}$
14. (Difficult) Factorise  $e^{2x} - e^x$
15. (Difficult) Factorise  $3x^2 + 17x + 10$

## Equations

16. Solve  $3x - 6 = 0$
17. Solve  $4 = 8x + 6$
18. Solve  $3x + 51 = 10x + 100$
19. Solve  $x^2 - 2x - 15 = 0$
20. (Difficult) Solve  $3x^2 + x + 2 = 0$

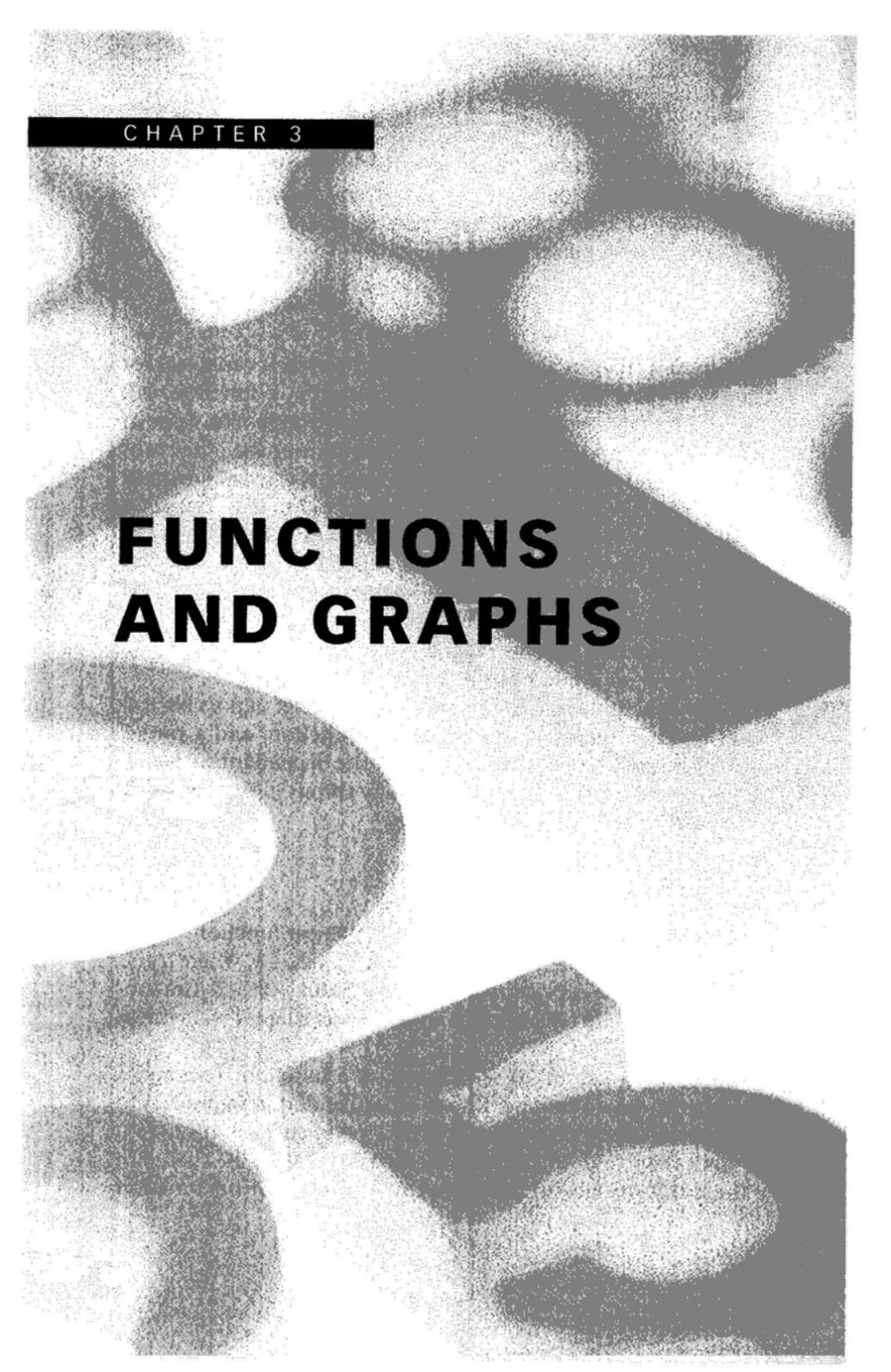
## Inequalities

21. Solve  $x + 2 < 3$
22. Solve  $-2x + 3 < 7$

## EXERCISES

**Systems of Equations**

23. Solve  $2x + 3y - 3 = 0$  and  $13 + 4y = 3x$
24. Solve  $2x - 3 = y$  and  $5y = 2x + 1$
25. Solve  $8x + 4y + 4 = 0$  and  $y = x + \frac{1}{2}$
26. (Difficult) Solve  $2x - 3 = y$  and  $2y = 4x - 6$
27. (Difficult) Solve  $x + 2y - 8 = 0$  and  $2x + 4y + 4 = 0$

The background of the page is a complex, abstract composition. It features several overlapping circles of varying shades of gray, creating a sense of depth and movement. In the lower right quadrant, there is a three-dimensional cube, also rendered in grayscale, which appears to be floating or resting on the surface. The overall effect is a high-contrast, textured visual field that serves as a backdrop for the chapter title.

CHAPTER 3

# **FUNCTIONS AND GRAPHS**

This chapter discusses functions, which are used to represent the relationships between economic and financial variables. You need to know about functions to understand economic and financial models.

## Functions

A **function** tells us about a relationship between variables. For a given value of the **independent variable**, a function will give us a unique value of the **dependent variable**. The value of the dependent variable depends on the value of the other variable, the independent variable. The way this is usually written is  $y = f(x)$  (pronounced  $y$  equals  $f$  of  $x$ ), and means that for a value of the independent variable (represented by  $x$ ) the function  $f$  will give us a value for the dependent variable (represented by  $y$ ). Note that the independent and dependent variables need not be  $x$  and  $y$  — we could have had  $w = f(u)$ , and other letters besides  $f$  can be used for the functional relationship — we could have written  $w = g(u)$ .

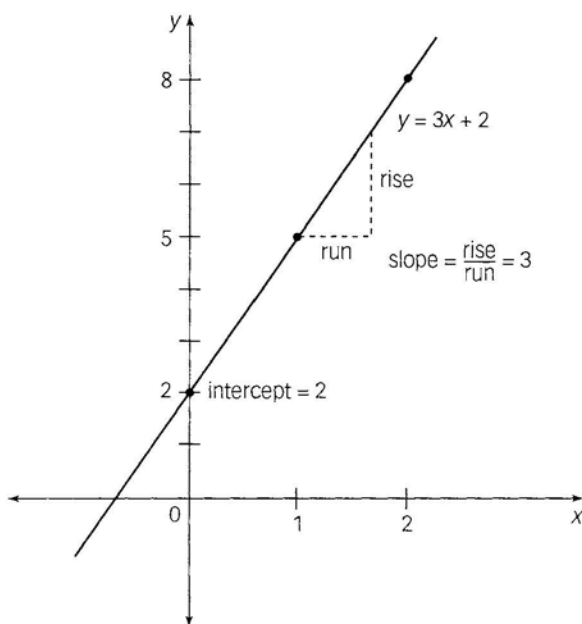
Any function has a **domain**, which is the set of possible values for the independent variable of a function, and a **range**, which is the set of possible values for the dependent variable.

To understand functions, like many things in mathematics, it is best to consider some examples.

## Linear Functions

The most common type of function is a linear function, which is one which can be written in the form  $y = ax + b$ , where  $a$  and  $b$  are constants. An example is the function  $y = 3x + 2$ . The supply and demand equations in the previous chapter were linear equations.

It is often helpful to see a graph of a function. To graph a function, draw a number line or axis (like that in figure 1.1) to represent the  $x$  values, and an axis perpendicular to this to represent the  $y$  values. The point marked 0 is the origin. Then plot the  $x$  and  $y$  values according to the function. For the function



**Figure 3.1** Linear function

$y = 3x + 2$ , if we pick  $x = 0$ , then  $y = 2$ ; if  $x = 1$ , then  $y = 5$ ; if  $x = 2$ , then  $y = 8$ ; and these values are plotted as in figure 3.1.

Fortunately there is an easier way of graphing a linear function than plotting points. If the equation is written in the form  $y = ax + b$  (called slope-intercept form), then the function is a straight line with a slope of  $a$  (slope is the vertical rise divided by the horizontal run) that crosses the  $y$ -axis at  $b$ . For  $y = 3x + 2$ , this means the line will have a slope of 3 and a  $y$ -intercept of 2. An intercept is the point where the line crosses the axis. The graph of this function is drawn in figure 3.1.

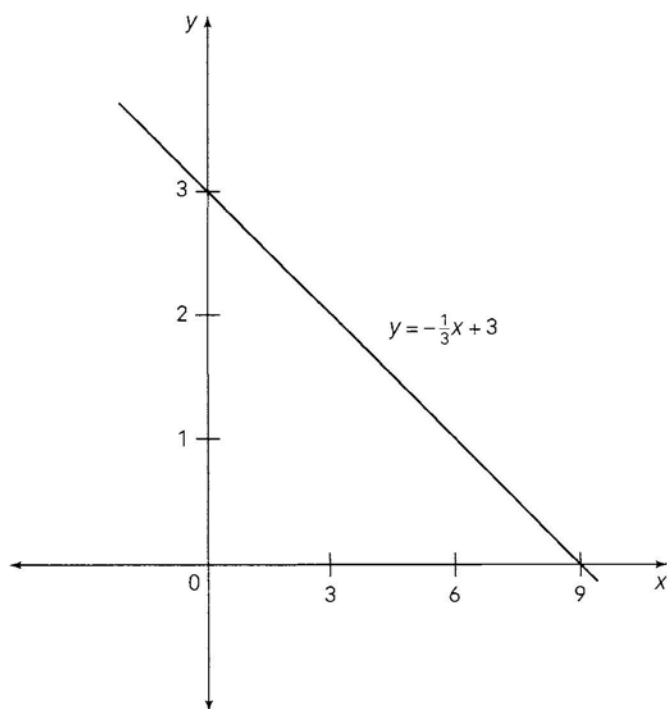
Some points need to be watched about graphing linear functions. First, we must write the function in exactly the form

$y = ax + b$  to be able to read off the slope and  $y$ -intercept. If we have  $x = -3y - 9$  then the slope is not 3; you must use the algebra skills in the previous chapter to rewrite it as

$$y = -\frac{1}{3}x + 3$$

and the slope is  $-\frac{1}{3}$  and  $y$ -intercept is 3. This is graphed in figure 3.2. Second, a negative slope means that the function slopes down to the right, as in figure 3.2, rather than up to the right. Third, the point where the graph of the function crosses the  $x$ -axis (the  $x$ -intercept) is the solution to the equation when  $y = 0$ . For the example we have just been considering, this is  $x = 9$ , as shown in figure 3.2.

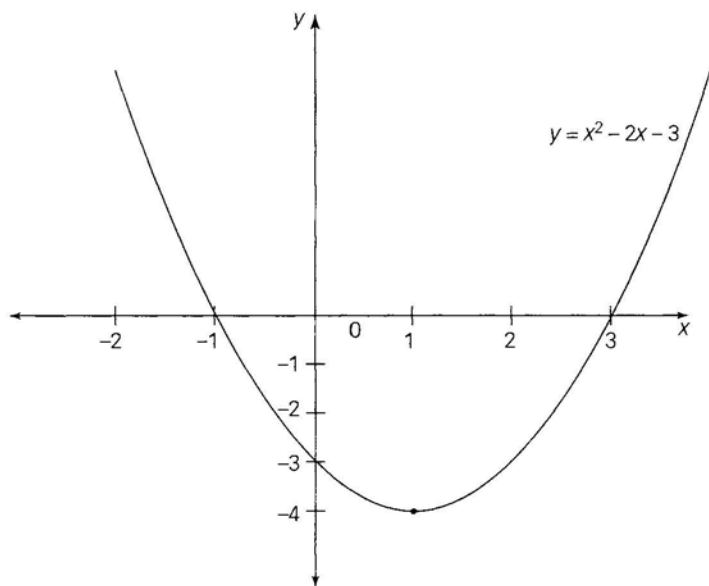
Note that  $y = b$  has a slope of 0 and a  $y$ -intercept of  $b$ , in other words it is a horizontal straight line through  $b$ . A vertical straight line has an equation  $x = c$ , and will go through  $c$ .



**Figure 3.2** Linear function with negative slope

## Quadratic Functions

Another type of function is the quadratic function, which can be written in the form  $y = ax^2 + bx + c$ . It is a nonlinear function because there are terms with higher powers of  $x$ . An example is  $y = x^2 - 2x - 3$ , which is graphed in figure 3.3.



**Figure 3.3** Quadratic function

When drawing the graph of a quadratic function the best way to proceed is to identify the intercepts. The  $y$ -intercept is easily obtained by substituting in  $x = 0$ , to give the  $y$  value. There will be two  $x$ -intercepts, which are the points where  $y = 0$ . These can be obtained by factorising the quadratic equation or by using the quadratic formula. This says that the  $x$  intercepts will be

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

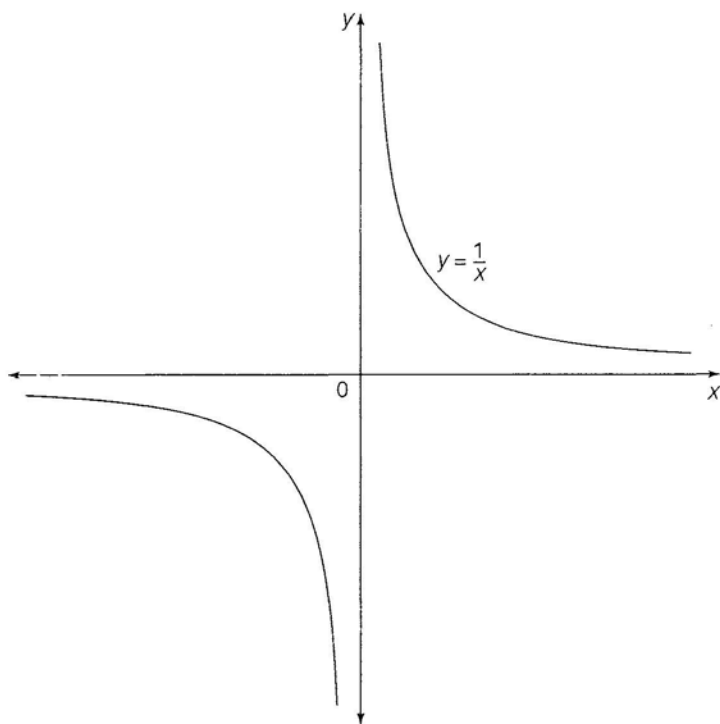
## Hyperbolic Functions

A hyperbola is a function which can be written in the form

$$y = \frac{a}{x} + b$$

The simplest example is  $y = \frac{1}{x}$

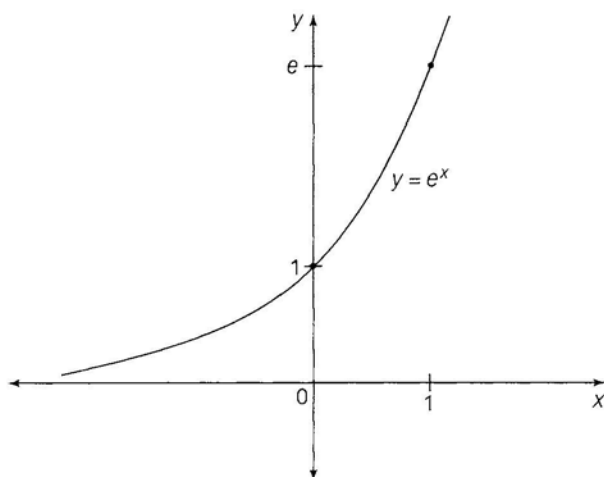
which is shown in figure 3.4. The hyperbola approaches the axes or asymptotes but never touches them.



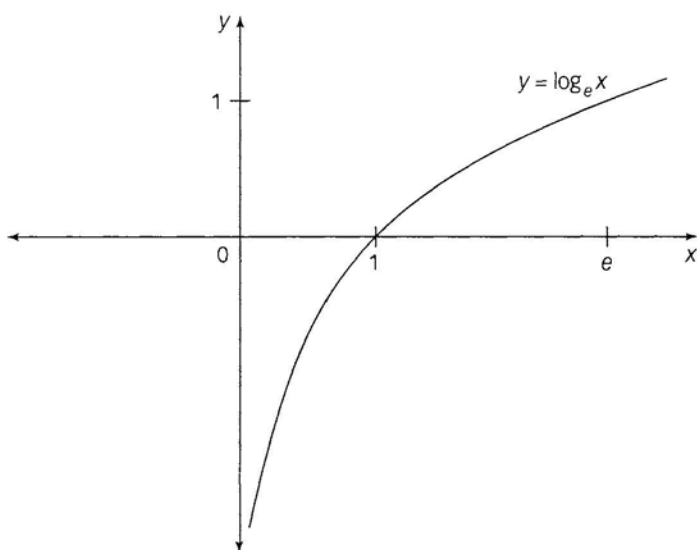
**Figure 3.4** Hyperbolic function

## Exponential and Logarithmic Functions

The exponential function  $y = e^x$  is shown in figure 3.5, and the logarithmic function  $y = \log x$  is shown in figure 3.6.



**Figure 3.5** Exponential function



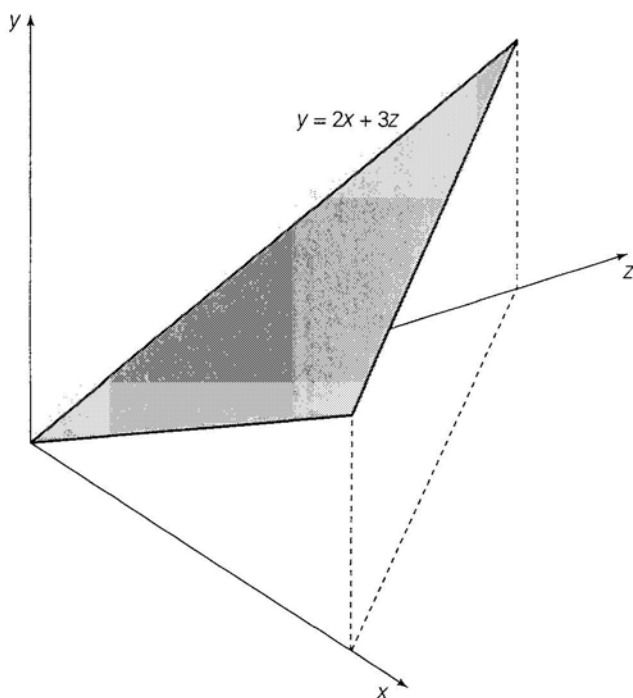
**Figure 3.6** Logarithmic function

## Polynomial Functions

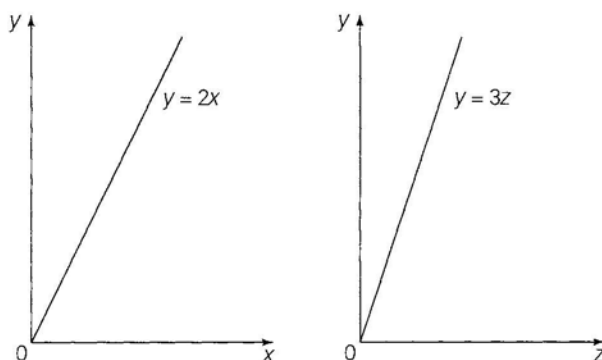
Polynomial functions have many terms and higher powers of  $x$ , for instance  $y = 5x^7 + 4x^5 - 8x^3 + 7$ . The graphs vary a lot in shape, so an example is not drawn.

## Functions with Many Variables

All the functions considered so far have only one independent variable,  $x$ , and a dependent variable,  $y$ . Many real-world relationships between variables are more complex than this, and we want to be able to represent them as functions. If  $y$  depends on both  $x$  and  $z$  we write  $y = f(x, z)$ . An example of such a function would be  $y = 2x + 3z$ .



**Figure 3.7** Linear function in two variables



**Figure 3.8** Linear function — partial curves

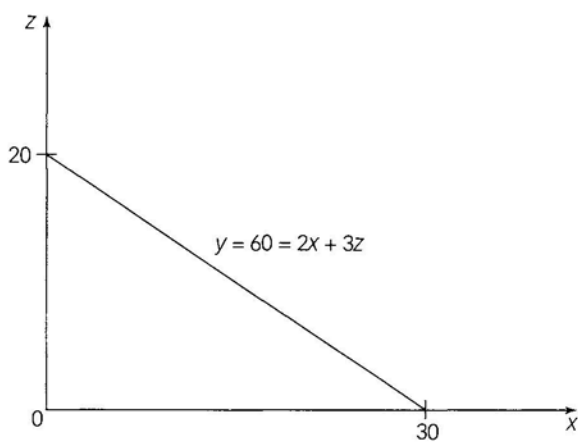
Graphing it is harder than the other functions we have considered so far because three variables cannot be easily drawn on a two-dimensional page. Nevertheless we can have a go, and  $y = 2x + 3z$  is the flat surface (called a plane) shown in figure 3.7.

There are other ways of graphing such functions that avoid the difficulty of drawing in three dimensions. One way is to choose a particular value of  $z$  and graph the relationship between  $y$  and  $x$ . Similarly, a particular value of  $x$  could be chosen and the relationship between  $y$  and  $z$  graphed. Figure 3.8 shows these partial graphs of the function.

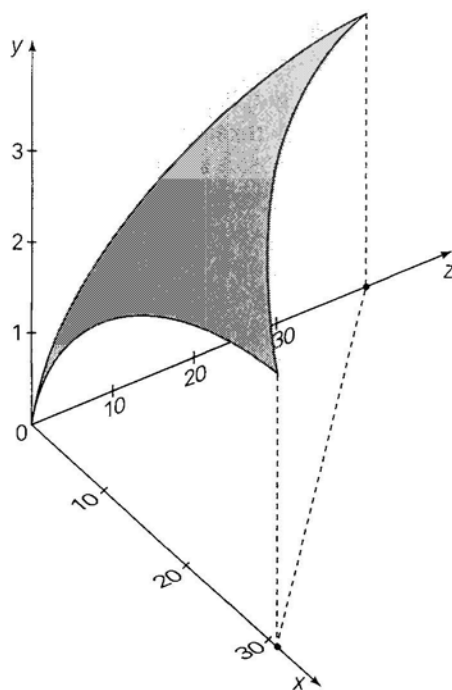
Another way is to choose a value for  $y$  and plot the  $x$  and  $z$  values which give this value of  $y$ . This is called a level curve of the function, and figure 3.9 shows the level curve of  $y = 2x + 3z$  when  $y = 60$ .

While these two-dimensional graphs of the function are easy to draw and work with, they each only partially represent the function. In introductory economics courses you will see these partial representations of the functions very often.

The function of several variables we considered was too simple to represent many relationships between economic variables. A common type of function you will encounter is the Cobb–Douglas function, named after two economists who used it to represent relationships between quantities of inputs and outputs in the production process. (It is a popular function



**Figure 3.9** Linear function — level curve



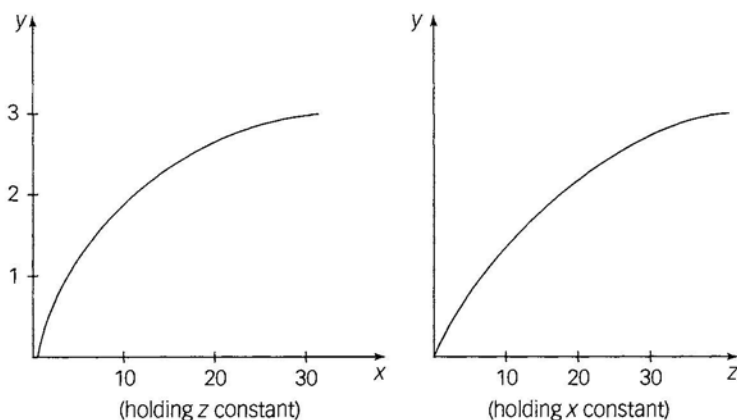
**Figure 3.10** Cobb-Douglas function

because it embodies many of the features economists assume about production processes, for instance positive but diminishing returns to using more of an input, diminishing marginal rate of substitution between inputs, and constant returns to scale. You may also see it used in economics courses to represent the utility individuals obtain from consuming goods.)

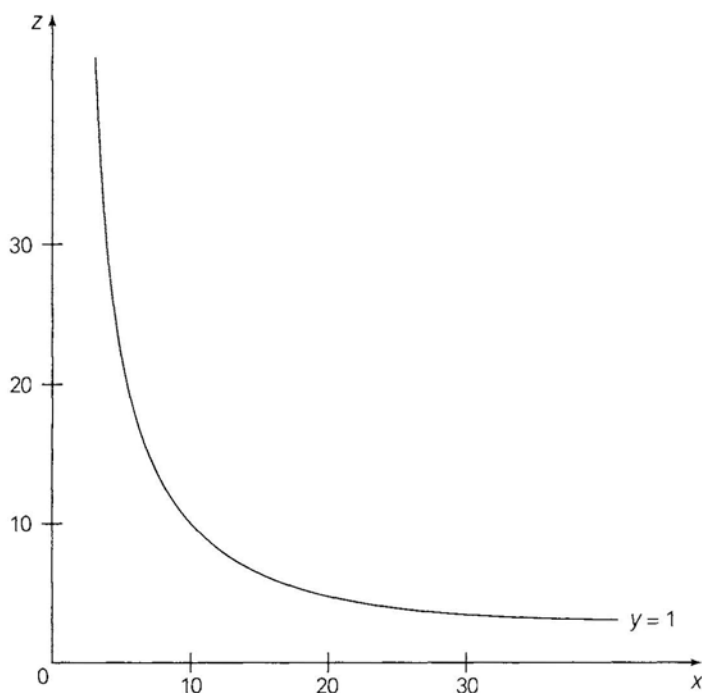
A Cobb–Douglas function has the form  $y = ax^bz^{1-b}$ , where  $a$  is a constant and  $b$  is a constant between 0 and 1. It is shown in figure 3.10. If it is a production function  $y$  is output and  $x$  and  $z$  are inputs; if it is a utility function  $y$  is utility and  $x$  and  $z$  are the quantities of goods consumed.

The partial graphs of the relationship between  $y$  and  $x$  holding  $z$  constant and the relationship between  $y$  and  $z$  are shown in figure 3.11. If we are talking about production then it shows what happens to output as we vary one of the inputs, holding the amount of the other input constant. This graph highlights the positive but diminishing returns that are a feature of a Cobb–Douglas production function. If we are talking about consumption, it indicates diminishing marginal utility.

Another way of looking at the function is to draw a level curve, which shows the values of  $x$  and  $z$  which yield a given



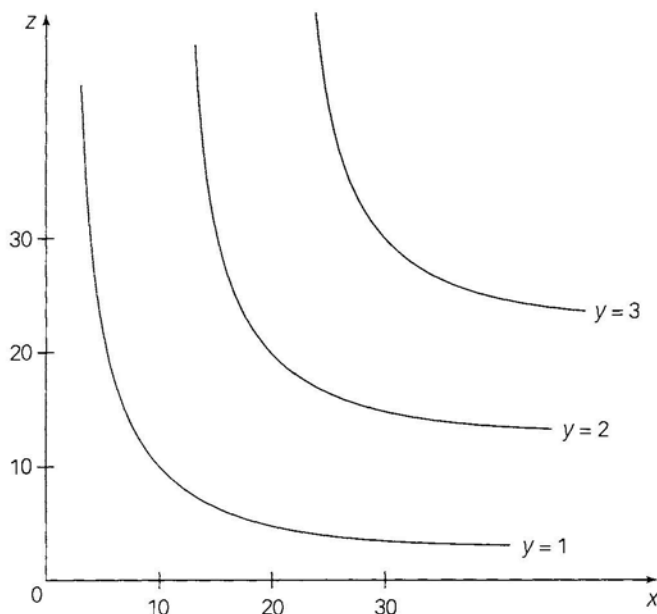
**Figure 3.11** Cobb–Douglas function — partial curves



**Figure 3.12** Cobb-Douglas function — level curve

level of  $y$ . A level curve for the Cobb-Douglas function at  $y = 1$  is shown in figure 3.12. In the economics of production, the level curve is called an isoquant and its shape highlights diminishing marginal rates of substitution between inputs; for consumption, the level curve is called an indifference curve and shows the diminishing marginal rates of substitution between goods.

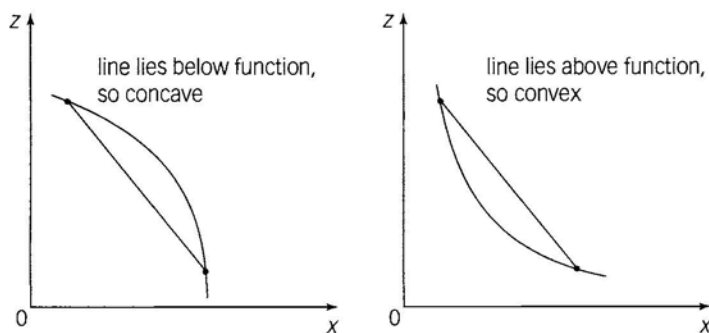
There are many level curves of this function, one for each level of  $y$  we could choose. For a Cobb-Douglas function all the level curves look the same as their distance from the origin is proportional to the level of  $y$  we choose. Figure 3.13 shows the level curves for  $y = 1$ ,  $y = 2$  and  $y = 3$ . In the economics of production this illustrates constant returns to scale.



**Figure 3.13** Cobb-Douglas function — many level curves

## Concavity and Convexity of Functions

An important property of functions that represent economic relationships is their concavity and convexity. It is really quite a simple idea. If you draw a line between two points on a function and this line always lies below or on the function, then the function is concave, while if it always lies above or on the function the function is convex. This is shown in figure 3.14. Look back over the functions introduced in this chapter — the logarithmic function is concave, and the hyperbola and exponential functions are convex. A linear function is both concave and convex. The idea applies to functions of several variables as well — the Cobb-Douglas function is concave, which by the way implies that the graphs of  $y$  against  $x$  or  $z$  are concave and that its level curves are convex.



**Figure 3.14** Concavity and convexity

## Inverse Functions

A question you may have is whether we can reverse a function and write  $x = f(y)$ . Sometimes this is possible, and the result is called the inverse of the function, but often the inverse function will not exist. The reason it may not exist is that the function specifies a unique  $y$  value for any given  $x$  value, but there may be many  $x$  values which yield the same  $y$  value. Thus any given  $y$  value will not yield a unique  $x$  value, which is what we need if the inverse function is to exist. Of the functions considered in the previous sections, linear functions will always have an inverse but many of the others do not. Exponential and logarithmic functions are inverses of each other.

**EXERCISES****Linear Functions**

1. Graph  $y = -2x + 3$  and indicate its slope and intercepts.
2. Write  $x + 2y - 8 = 0$  in slope-intercept form, and graph it indicating its slope and intercepts.
3. Graph  $y = 3$  and indicate its slope and intercepts.
4. Graph  $x = 3$  and indicate its slope and intercepts.
5. Work out the equation of the straight line passing through the points  $x = 1, y = 1$  and  $x = 2, y = 4$ .

**Quadratic Functions**

6. Graph  $y = x^2 + 9$ .

**Exponential and Logarithmic Functions**

7. Graph  $y = 2e^x$ .

**Functions of Several Variables**

8. (Difficult) For the function  $U = AB$ , draw the partial plots and the level curves.



CHAPTER 7

# FINANCE

This chapter covers most of the mathematics used in introductory accounting and finance courses. It will be useful outside your studies as you consider investments, superannuation and housing loans.

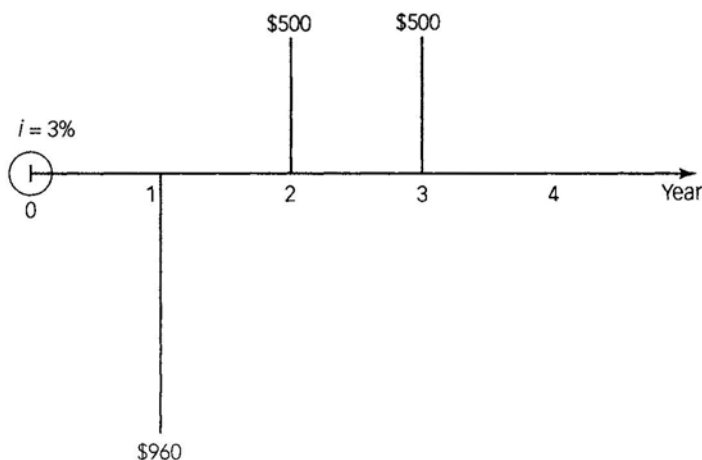
## The Time Value of Money

The need for the mathematics in this chapter comes from the existence of **positive rates of interest**. If interest rates are positive, the value of a dollar depends on when it is received. A dollar received now will be worth more than a dollar to be received in a year's time. Much of what the finance industry does is take advantage of this and trade dollars across time (as well as trading dollars with different degrees of riskiness).

A point which often confuses students is the relationship between positive rates of interest and positive rates of inflation, so let's be clear about this at the beginning of the chapter. A positive rate of inflation means the general price level is rising and is another reason, besides a positive rate of interest, for a dollar now to be worth less than a dollar in a year's time. Both inflation and interest make dollars at different dates worth different amounts. The concept of the **real rate of interest** separates the two effects. The real rate of interest is equal to the nominal rate of interest (the rate we see advertised by borrowers and lenders) less the rate of inflation. For instance, if the nominal interest rate is 8% and the inflation rate is 2%, then the real interest rate is 6%. It isolates the true time value of money component of the interest rate, net of the component of the interest rate that is compensating for inflation. Real rates of interest tend to be more constant over time and across countries than the nominal interest rates. When using the mathematics in this chapter to make financial decisions, it will usually be best to remove the inflation component and work with real interest rates.

Having clarified this point, let's now look at diagram which will help us to value dollars at different dates.

Figure 7.1 is a time line marked with years from the present time, marked zero. Cash inflows in the future are shown above the line at the dates they occur, and cash outflows are shown



**Figure 7.1** Representing cash inflows and outflows

below the line at the dates they occur. Reading the figure, we will outlay \$960 in cash a year from now and receive \$500 at the end of each of the subsequent two years. In order to value this stream of cash flows we have to know the date they need to be valued and the interest rate per year. We will assume we need to know the value now (or present value) of the stream and indicate this on the figure by circling that date. We will assume an interest rate of 3%. Interest is compounded, which means that further interest is earned in future periods on interest earned in past periods.

How do we value this stream of payments? Consider how much a dollar now is worth in a year's time at an interest rate of 3%. It will be worth  $\$1.00 \times 1.03 = \$1.03$ , so a dollar now is worth  $\frac{100}{103}$  or approximately .97 of what a dollar in a year's time is worth. This means that the \$960 outflow at the end of the first year is worth \$932. By similar reasoning a dollar at the end of two years is worth  $\$1.00 \times 1.03 \times 1.03 = \$1.06$  and a dollar now is worth approximately .94 of what a dollar in two year's time is worth, and the \$500 inflow is worth \$471. Again, by similar reasoning, a dollar at the end of three years is worth approximately \$1.09. A dollar in three years' time is worth approximately .92 of what a dollar now is worth. The \$500 outflow at the end of three years is worth \$458. The net

value in current period dollars (or net present value) of the cash outflows and inflows when the interest rate is 3% is  $-\$932 + \$471 + \$458 = -\$3$ . Thus if you were offered these cash flows as an investment proposition you should not accept it. Note that if you had not considered the time value of money you may have accepted it since  $-\$960 + \$500 + \$500 = +\$40$ .

There are formulas which make calculations like those in the previous paragraph easier. The **future value** in  $n$  years' time of a cash inflow or outflow of an amount of  $\$A$  now, at an interest rate of  $i\%$ , is  $F = A(1 + i)^n$ . (Note that when we write  $i\%$  this means  $i/100$ , so that  $i = 3\%$  means  $i = \frac{3}{100} = .03$ .) The formula for the **present value** of  $\$A$  in  $n$  years is

$$P = \frac{A}{(1 + i)^n} \text{ or equivalently } P = A(1 + i)^{-n}$$

These formulas for the future and present values are really the same, it is just that putting  $(1 + i)$  to the power of positive  $n$  moves the valuation date of the present cash flow forward in time — i.e. finds the future value — while putting  $(1 + i)$  to the power of negative  $n$  moves the valuation date of the future cash flow backward in time — i.e. finds the present value. Variations of this method of valuing payments at different dates will be applied to many types of financial problems in this chapter.

Before moving on to these problems, some comments about the time period over which interest is compounded. In the above illustration and formulas the period was a year, but frequently in financial transactions this is not the case. Interest may be calculated quarterly, monthly or daily, or even continuously. The frequency of calculation may differ from the frequency of payment, for instance it may be calculated daily and paid quarterly, but it is the frequency of calculation which matters. Fortunately the formulas given in the previous paragraph apply irrespective of the frequency of calculation, provided we interpret  $i$  as the interest rate per period and  $n$  as the number of periods. For instance if the interest rate is 12% per annum, calculated monthly, and the term of the investment was two years, then  $i = 1\%$  and  $n = 24$ . Always be careful that the time interval used to work out  $i$  and the time interval used to work out  $n$  are the same.

Does the frequency of compounding make a difference? It certainly does, because the more frequently it is compounded the more interest is earned on interest. To see this, compare the value at the end of two years of \$1 under yearly compounding with the value at the same interest rate and monthly compounding. For yearly compounding  $i = 12\%$  and  $n = 2$ , giving a value of \$1.25. For monthly compounding  $i = 1\%$  and  $n = 24$ , giving \$1.27. More frequent compounding thus works in favour of the investor.

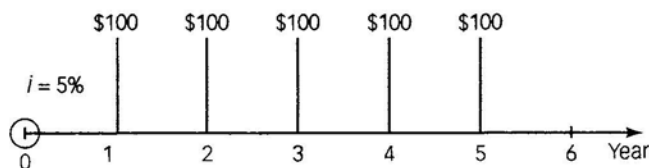
This issue of the frequency of compounding creates a problem of comparing interest rates calculated at different frequencies. Usually this problem is solved by converting the rate to an **effective annual rate**, which is the annually compounded rate that would yield the same return. If we have a rate of  $i\%$  per annum compounded  $m$  times per year the effective annual rate is

$$r = \left(1 + \frac{i}{m}\right)^m - 1$$

Another interest rate concept you should know about is **continuous compounding**. This is the limiting case of frequent compounding, and while it is obviously impossible in practice to compound many times a second, mathematicians have worked out what such compounding would amount to by choosing larger and larger numbers for  $m$  in the last equation and observing that  $r$  approaches a certain value. This value is given in the following equation. If the interest rate is  $i\%$  compounded continuously, then the effective annual rate is  $r = e^i - 1$ , where  $e$  is the natural number discussed in chapter one.

## Annuities and Perpetuities

The discussion of the time value of money in the previous section was restricted to examples where there was just one future payment to be valued. A more common situation is a stream of future payments, called an **annuity**. An example of an annuity would be a pension of \$100 paid each year for 5 years after retirement, as illustrated in figure 7.2.



**Figure 7.2** Present value of an annuity

The present value of this annuity could be calculated by applying the formula in the previous section to each payment, but this becomes tiresome when there is a large number of payments. Instead you can use the following formula for the present value of an annuity of \$ $A$  paid at the end of every  $n$  years when the interest rate is  $i\%$ . It is

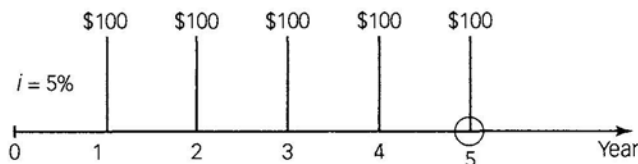
$$P = \frac{A[1 - (1 + i)^{-n}]}{i}$$

For our example,  $n = 5$ , and let's assume  $i = 5\%$ , so the present value of \$100 payments is

$$P = \frac{1 - (1.05)^{-5}}{.05} = \$433$$

This is the amount you should be prepared to pay to purchase the annuity.

A slightly different problem would be to work out the future value of the annuity, which is the accumulated amount at the end of the annuity if payments received are invested at the interest rate. A time line of this problem is shown in figure 7.3.



**Figure 7.3** Future value of an annuity

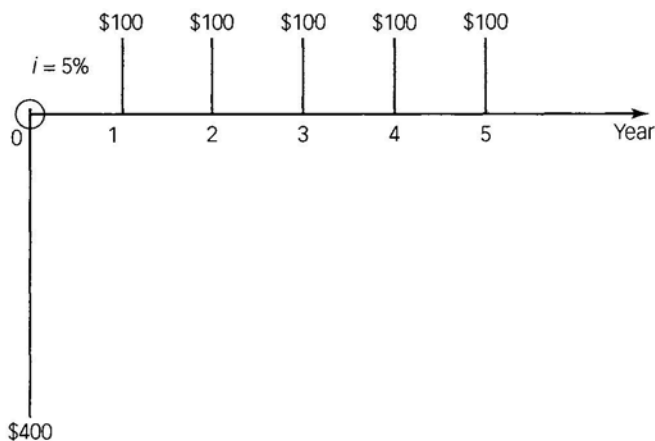
Notice that the only difference between this and the previous figure is the different valuation date. This suggests the easiest way to solve the problem, which is to take the present value of the annuity we have just calculated, and, treating this value as a single payment, value it at the end of the annuity using the formula for a single payment. In our example the present value of the annuity was \$433 and we find the value of this in five years' time by multiplying it by  $(1 + i)^5$ , giving  $P = 433(1.05)^5 = \$553$ .

A problem you will often encounter is valuing a stream of payments that extend an infinite time into the future, called a **perpetuity**. A formula for the present value of a perpetuity has been worked out by mathematicians choosing larger and larger numbers for  $n$  in the formula for the present value of an annuity and seeing what amount the value of the annuity approaches. The value of a perpetuity of \$ $A$  at an interest rate of  $i\%$  is  $P = A/i$ . For example if our \$100 pension payments were for an infinite period their value would be  $P = 100/.05 = \$2000$ . Most new students of finance are surprised that perpetuities are not worth more, but remember that payments a long way into the future are heavily discounted and thus worth very little.

A final note before we move on to some specific financial problems. In many accounting and finance textbooks you will find tables showing the present and future values of single amounts and annuities. Often you will find it easier to use these tables than the above formulas in calculating values and interest rates, but make sure you understand the idea of the formulas, and that you get the right table and use it correctly. Also be sure to check whether or not tables can be used in examinations in your course.

## Investments

A common problem faced by firms and others is evaluating possible investments. For instance if the firm requires a return of 5% on funds invested and is evaluating an investment with an initial outlay of \$400 and yield profits of \$100 per year for the next 5 years, should it make the investment? The question



**Figure 7.4** Net present value of an investment

can be answered by calculating the net present value of the investment. The net present value of an investment is much the same thing as the present value of an annuity, the net just means that we have to subtract the initial outlay from the present value of the stream of profits.

First, and I suggest you do this before trying to solve any problem, draw a time line to represent the investment. The time line is figure 7.4. In the previous section we calculated the present value of the stream of 5 payments of \$100 at 5% was \$433, so the net present value of the investment is  $\$433 - \$400 = \$33$ . Since the net present value is positive, the firm should undertake the investment. Under the conditions we are dealing with (i.e. the future outlay, returns and rates are known with certainty) the rule is that the firm should invest if the net present value is positive and not invest if it is negative.

The problem we have just considered is very simple but the rule applies to situations where the yearly profits are of unequal amounts, irregularly timed, where there are additional outlays in later periods, where the investment has a scrap value, and many other situations. Of course the calculations of the present values in these situations will be more complex, but they can be done by breaking down the complex streams of payments into components that can be valued with

the formulas you know for the values of single payments and annuities, making sure they are all valued at the present.

An alternative to the net present value rule for evaluating investments is the **internal rate of return rule**. This says that you should invest if the internal rate of return for the project exceeds the required rate of return.

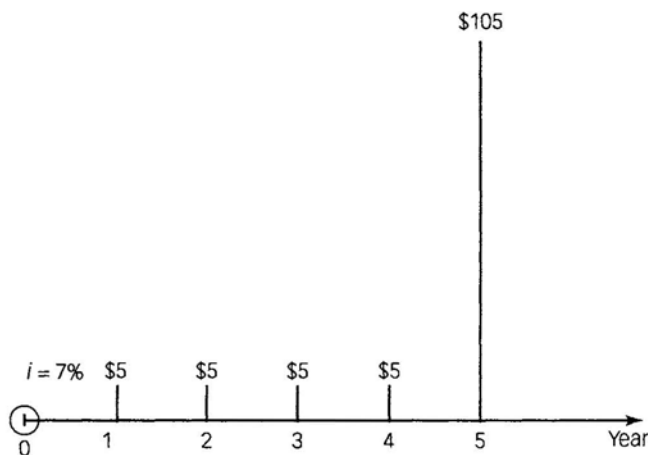
The internal rate of return is defined as the rate which would make the net present value of the project equal to zero. How is it calculated?

Consider the example above, which has an initial outlay of \$400 and yearly profits of \$100 for 5 years. We are looking for the  $i$  which makes the net present value zero, which is the  $i$  which solves  $0 = 400 - 100[1 - (1 + i)^{-5}]/i$ . The expression on the right of the equation is the same as the one we used above to calculate the net present value of the investment. Solving this equation for  $i$  is not easy even for this simple example, so you will often have to resort to trial and error. Pick a reasonable value for  $i$  and evaluate the right-hand side of the equation. If the value is positive, pick a larger  $i$  and try again; if it is negative, pick a smaller  $i$  and try again. Eventually you will have an  $i$  which gives approximately zero, and this is your internal rate of return. In this example the internal rate of return is a little over 5%.

## Bonds

A **bond** or **debenture** from the point of view of an investor is a purchase price which must be paid now, a redemption or maturity value which will be received at the end of the term of the bond, and usually some interest that will be received during the term of the bond.

An investor or bond trader who knows these details and the yield they require often wants to calculate how much to pay for the bond or the **purchase price**. Let's see how this is done. First show the payments and receipts on a time line. Let's say the required yield is 7%, the maturity value is \$100, and there is annual interest of \$5 over the five-year term of the bond. (This would mean that the so-called coupon rate used to calculate the interest payments based on the maturity value of the



**Figure 7.5** Purchase price of a bond

bond of interest is 5% — it is important not to confuse this coupon rate with the yield on the bond.) The time line for the bond is shown in figure 7.5.

The amount that should be paid for the bond is the present value of the interest receipts and maturity value. To value the interest receipts, recognise they are an annuity, and use the annuity formula above with a term of 5 years, an interest rate of 7% and amount of \$5, giving us

$$P = \frac{A[1 - (1+i)^{-n}]}{i} = \frac{5[1 - (1.07)^{-5}]}{0.07} = \$20.50$$

The maturity amount is a single payment so use that formula with an amount of \$100, a term of 5 years and an interest rate of 7%, giving us  $P = A(1+i)^{-n} = 100(1.07)^{-5} = \$71.30$ . The amount that should be paid for the bond is thus  $\$20.50 + \$71.30 = \$91.80$ .

Notice that the purchase price is less than maturity value, and this will always be the case if the coupon rate is less than the yield. In the quite common case of a bond which does not have interest payments during the term (which can be viewed as a coupon rate of 0%) the purchase price will always be less than the maturity value.

Also note that the higher the yield the lower the price. For instance if the required yield was 10% then the present value of the interest payments would be  $P = A(1 + i)^{-n} = 100(1.10)^{-5} = \$62.09$  and the present value of the maturity amount would be  $P = A[1 - (1 + i)^{-n}]/i = 5[1 - (1.10)^{-5}]/.10 = \$18.95$ , giving a price of \$81.04. This is less than the \$91.80 price for a yield of 7%.

The other thing an investor or bond trader may want to know is the **yield** on the bond for a given purchase price. This is like calculating the internal rate of return on an investment — you will often have to resort to trial and error with different interest rates. In simple cases it may be possible to solve the equation for the purchase price of the bond for the interest rate. We will look at a simple case of a bond with a purchase price of \$95 and 2 years to maturity with no interest payments. The equation for the present value of a single payment was  $P = A(1 + i)^{-n}$ , and rearranging this gives

$$i = \left( \frac{A}{P} \right)^{1/n} - 1$$

Substituting gives

$$i = \left( \frac{100}{95} \right)^{1/2} - 1 = .026 = 2.6\%$$

## Shares

Shares are more difficult to value than bonds because the amount and timing of the dividend receipts are uncertain. They also have no maturity value, but instead may be sold at some time in the future for an uncertain price.

The valuation of **uncertain streams of future payments** (including shares, options, futures contracts) is an advanced topic beyond the scope of this book. Part of the complexity comes from the way that holding portfolios of several shares with uncertain streams of future payments modifies the overall level of uncertainty of the portfolio.

However, the mathematical tools outlined in the previous sections can be used to value single shares and calculate yields

on shares if you are prepared to guess the timing and amounts of the future payments.

## Depreciation

A common calculation you will have to carry out in accounting courses is depreciation on assets. As you will discover in your accounting courses, depreciation is not an attempt to estimate the value of an asset but to **allocate the cost of an asset over the useful life of the asset** for the purposes of calculating accounting profit.

There are two main methods of depreciation — the straight-line method and the diminishing value method. The straight-line method allocates the cost of the asset equally over the useful life of the asset while in the diminishing value method more of the cost is allocated to the early part of the life of the asset.

Under the **straight-line method** the depreciation rate for an asset with a useful life of  $n$  years is  $\frac{1}{n}$ , and this rate is applied to the difference between the cost  $C$  of the asset and the scrap value  $S$  to give the depreciation charge each year. The depreciation charge will be the same each year. The formula for the depreciation charge each year is

$$D = \frac{C - S}{n}$$

For instance, if an asset has a cost of \$80, a scrap value of \$20 and a useful life of three years, the depreciation rate is

$$r = \frac{1}{n} = \frac{1}{3} = 33\%$$

and the depreciation charge each year will be

$$D = \frac{C - S}{n} = \frac{80 - 20}{3} = \$20$$

The book value, which is the cost less accumulated depreciation, will be \$60 at the end of the first year, then \$40, then the scrap value of \$20 at the end of the third year. The depreciation table is shown in figure 7.6.

Under the **diminishing value method**, the rate  $r$  is applied to the book value of the asset rather than the cost of the asset.

**Cost = \$80, Depreciation rate = 33% straight line**

<i>Year</i>	<i>\$ Depreciation charge</i>	<i>\$ Accumulated depreciation</i>	<i>\$ Book value at end of year</i>
1	20	20	60
2	20	40	40
3	20	60	20

**Figure 7.6** Straight-line depreciation table

Since the book value falls over the life of the asset, the yearly depreciation charge will also fall over the life of the asset. For instance, if the asset in the above example was depreciated at a rate of 33% diminishing value, the depreciation charge in the first year would be the book value of \$80 multiplied by the rate .33 giving \$26.40. In the second year the book value will be  $\$80 - \$26.40 = \$53.60$ , so the depreciation charge will be \$17.69. The next year the book value will be  $\$53.60 - \$17.69 = \$35.91$  and the depreciation charge will be \$11.97. The depreciation table is shown in figure 7.7.

**Cost = \$80, Depreciation rate = 33% diminishing value**

<i>Year</i>	<i>\$ Depreciation charge</i>	<i>\$ Accumulated depreciation</i>	<i>\$ Book value at end of year</i>
1	26.40	26.40	53.60
2	17.69	44.09	35.91
3	11.97	56.06	23.94
4	7.90	63.96	16.04
5	5.29	69.25	10.75

**Figure 7.7** Diminishing value depreciation table

In general, for the diminishing value method, the book value at the start of year  $n$  of an asset costing  $C$  with a depreciation rate of  $r$  will be  $B = C(1 - r)^{n-1}$  and the depreciation charge in year  $n$  will be this times  $r$ , which is  $D = rC(1 - r)^{n-1}$ .

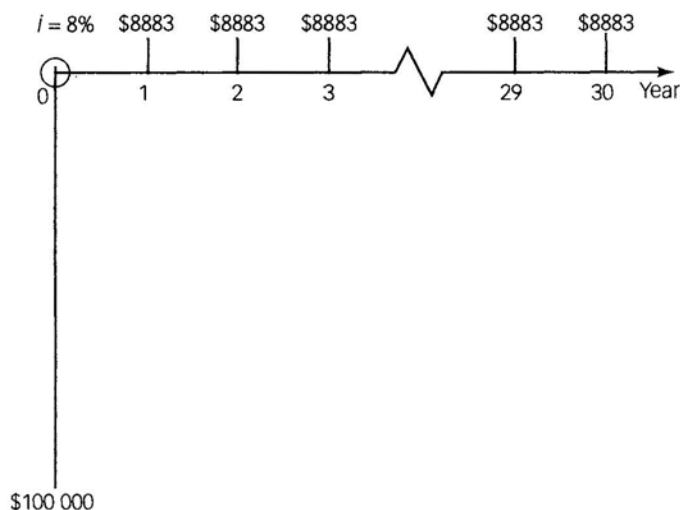
In the above examples the depreciation rate  $r$  was known, but what happens if we need to calculate it? Given that the

useful life of the asset is  $n$  years, the cost is  $C$  and the scrap value is  $S$ , then

$$r = 1 - \left( \frac{S}{C} \right)^{1/n}$$

## Loans

A special feature of home loans and many other loans is that the monthly repayment is fixed over the whole term of the loan. The cash flows are illustrated in figure 7.8.



**Figure 7.8** Home loan

A common problem is determining the annual repayment amount  $A$  for a loan of amount  $P$  over a period  $n$  years. This is solved by recognising that the series of repayments is an annuity which can be valued using the formula for the present value of an annuity, i.e.

$$P = \frac{A[1 - (1+i)^{-n}]}{i}$$

Here we know the loan amount  $P$ , the term  $n$  and the interest rate  $i$ , and are trying to calculate  $A$ . For instance, if the interest rate is 8% per annum and you want to borrow \$100 000 over 30 years, what is the annual repayment? Rearranging the annuity formula we have

$$A = \frac{Pi}{1 - (1+i)^{-n}}, \text{ so } A = \frac{100\,000 \times .08}{1 - (1.08)^{-30}} = \$8883$$

If the problem was slightly different and repayments were monthly at an interest rate of 8% per annum calculated monthly, then  $n = 360$  and  $i = .666\%$ , so

$$A = \frac{100\,000 \times .00666}{1 - (1.006\,66)^{-360}} = \$734$$

This will be  $\$734 \times 12 = \$8808$  per year. Notice how making more frequent payments helps the borrower.

Another common problem is to work out how much of each repayment is interest and how much is repayment of principal. The loan repayment table in figure 7.9 gives us this information for the loan with yearly repayments.

Year	\$ Loan outstanding at start of year	\$ Repayment	\$ Interest	\$ Principal	\$ Loan outstanding end of year
1	100 000	8 883	8 000	883	99 117
2	99 117	8 883	7 929	954	98 163
3	98 163	8 883	7 853	1 030	97 133
↓					
30	8 225	8 883	658	8 225	0

**Figure 7.9** Home loan repayment table

The loan amount is \$100 000 and the yearly repayment we calculated above was \$8883. In the first year, the interest charge will be  $\$100\,000 \times .08 = \$8000$ , so the remainder of the repayment  $\$8883 - \$8000 = \$883$  is principal. The principal at the end of the first year is  $\$100\,000 - \$883 = \$99\,117$ , and this

becomes the principal at the start of the second year. The rest of the repayment table is filled out in a similar manner. Notice that the proportion of the yearly repayment eaten up in interest declines as the years go on, or, looking at it another way, you repay the principal more quickly as the loan goes on. The loan outstanding at any time can be read off the table, and this enables the effect of changes in interest rates or the term of the loan to be calculated.

## EXERCISES

**Interest**

1. If the nominal interest rate is 18% and inflation is 3%, what is the real rate of interest?
2. Draw a time line to represent \$100 invested at 7% per annum for 15 years. What amount will it accumulate to? Verify that the present value of this accumulated amount is \$100.
3. Calculate the effective annual rate for 8% per annum calculated yearly, quarterly, weekly and continuously.

**Annuities and Perpetuities**

4. Calculate the present value of a pension of \$20 000 per year if the interest rate is 3% and you expect to live for another 20 years. This is the lump sum received now which would be equivalent to the pension. What happens to the present value of the pension if the interest rate rises? What happens if you expect to live longer than 20 years? How much would this pension paid for ever be worth?

**Investments**

5. Draw a time line to represent an investment with an initial outlay of \$200, and receipts of \$30 at the end

## EXERCISES

of the first year, \$40 at the end of the second and third years, and \$150 at the end of the fourth year. Calculate the net present value of the investment at 6%, 8% and 10%. Now consider another investment with the same initial outlay and receipts of \$80 at the end of the first year, \$70 at the end of the second year, \$60 at the end of the third year, and \$35 at the end of the fourth year. Calculate the net present value of this investment at 6%, 8% and 10% and compare them to the other investment. Why does the choice of the best investment depend on the interest rate? Based on your calculations, guess the internal rates of return on the two projects.

6. Draw a time line to represent an investment with an initial outlay of \$200, another outlay of \$20 at the end of the fourth year, and receipts of \$70 at the end of the first, second, fifth and sixth years. Calculate the net present value at 5%.

**Bonds**

7. Find the price of a bond with a maturity value of \$100, coupon interest of \$3 at the end of each year, and a term of 10 years. The required return is 7%.

## EXERCISES

8. (Difficult) Calculate the price of the bond in the previous question at the beginning of the fifth year. What would the bond be worth at the beginning of the fifth year if interest rates rise at that time to 10%? What would the present value of the bond have been at the beginning of the first year if this interest rate change had been anticipated?

## Depreciation

9. Prepare a straight line depreciation table for an asset with a cost of \$500, a scrap value of \$100 and a rate of  $33\frac{1}{3}\%$ .
10. Prepare a diminishing value depreciation table for this asset at a rate of 41%. Verify using the formula given in the chapter that 41% is the rate required to depreciate the asset over its useful life of three years.

## Loans

11. Prepare a loan repayment table for \$150 borrowed for three years at an interest rate of 7% per annum. You will need to calculate the annual repayment amount.

## EXERCISES

12. (Difficult) For the loan in the previous question, let's say interest rates rise to 9% at the end of the first year. What is the new annual repayment to still repay the loan over 3 years? How much would a borrower be willing to pay at the end of the first year to avoid the interest rate increase?

# Solutions to exercises

## CHAPTER 1

1. 49
2. 36
3.  $\frac{4}{9}$
4. -2
5. 10
6. 4
7. 4
8.  $\frac{1}{4}$
9. 5
10. 16
11. 1296
12. 196
13.  $\frac{4}{9}$
14. 113
15. 5

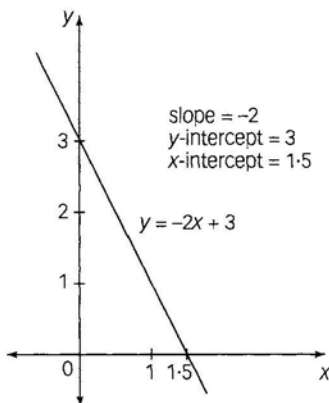
## CHAPTER 2

1.  $-27x - 6$
2.  $x^2 + 6x + 9$
3.  $10x^2 + 23x - 21$
4.  $5x^2 + 5x - 3$
5.  $3x^2 + 7x - 2x^2 + x - 6 = x^2 + 8x - 6$
6.  $\frac{(x+3)(x-1)}{(x+3)(x+1)} = \frac{x-1}{x+1}$
7.  $\frac{(x+2)(x-1)}{(x+2)(x+6)} = \frac{x-1}{x+6}$

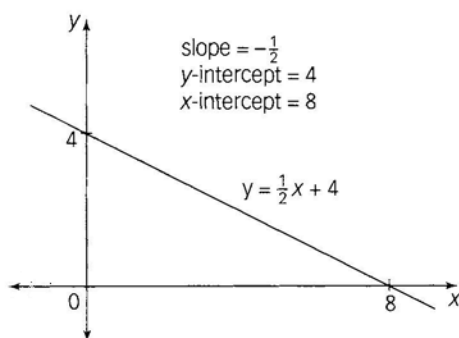
8.  $\frac{7x}{12}$
9.  $\frac{6x-7}{21x} = \frac{2}{7} - \frac{1}{x}$
10.  $(x-1)(x+6)$
11.  $(2x+1)(x-3)$
12.  $(x-5)(x+5)$
13.  $x^{1/3}(4-x^{1/3})$
14.  $e^x(e^x-1)$
15.  $(3x+2)(x+5)$
16.  $x=2$
17.  $x=-\frac{1}{4}$
18.  $x=-7$
19.  $x=5$  or  $x=-3$
20. There is no  $x$  value which solves this equation.
21.  $x < 1$
22.  $x > -2$
23.  $x=3$  and  $y=-1$
24.  $x=2$  and  $y=1$
25.  $x=-\frac{1}{2}$  and  $y=0$
26. There is an infinite number of solutions. The two equations are identical.
27. There are no  $x$  and  $y$  values which solve this equation.

**CHAPTER 3**

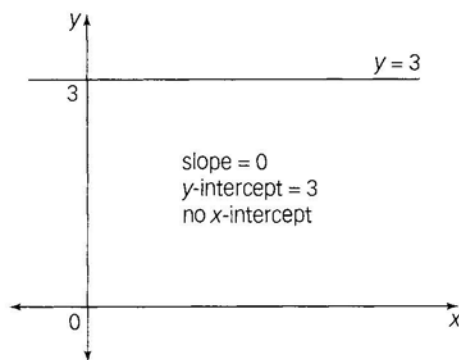
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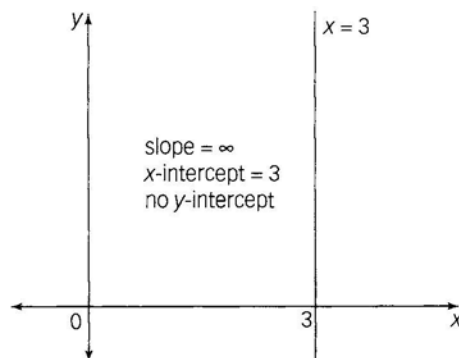
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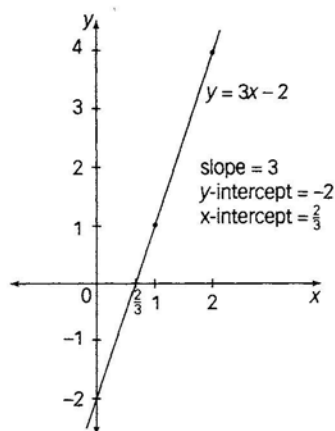
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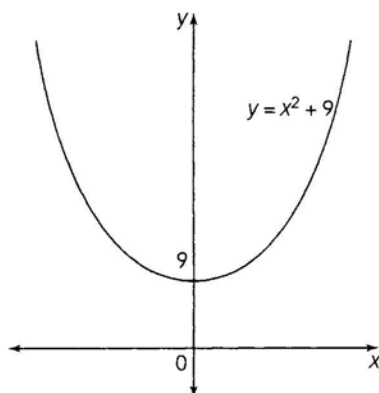
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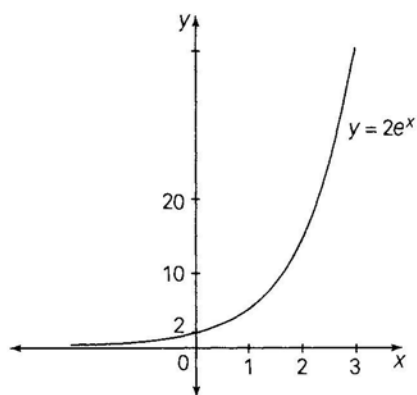
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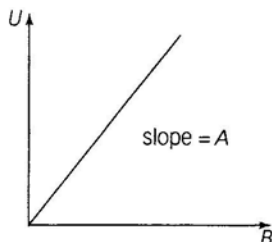
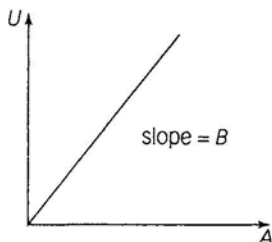
6.



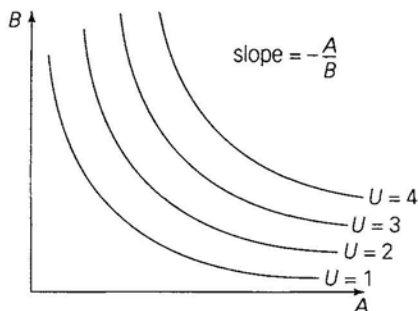
7.



8. Partial plots of  $U = AB$



Level curves of  $U = AB$



#### CHAPTER 4

1.  $8x^7$
2.  $2x^{-2}$
3.  $6x + 6$
4.  $\frac{1}{2}x^{-1/2}$
5.  $3(6 - x) - 1(3x + 6) = 12 - 6x$
6.  $-\frac{1}{(x-1)^2}$
7.  $3 - 7x^{-2}$
8.  $\frac{8(x^2+1)^2 - 4x(x^2+1)8x}{(x^2+1)^4} = \frac{8-24x^2}{(x^2+1)^3}$
9.  $2 \times 3x^2(x^3-1)^4 = 6x^5 - 6x^2$
10.  $x(x^2-1)^{-1/2}$
11.  $2(x + \frac{1}{x})(1 - \frac{1}{x^2}) = 2(x - \frac{1}{x^3})$
12.  $2e^{2x}$

## CHAPTER 7

1. 15%

$$2. F = P(1+i)^n = 100(1.07)^{15} = \$275.90. P = F(1+i)^{-n} = 275.90(1.07)^{-15} = \$100.$$

3. 8%, 8.24%, 8.32%, 8.33%

$$4. P = \frac{[A(1-(1+i)^{-n})]}{i} = \frac{20\,000[1-(1.03)^{-20}]}{.03} = \$297\,549$$

$$P =$$

If  $i$  rises,  $P$  falls. If  $n$  increases,  $P$  rises. If paid forever,

$$P = \frac{20\,000}{.03} = \$666\,666$$

5. NPV at 6% is \$16.30, 8% is \$4.08, 10% is approximately zero. For the other investment NPV at 6% is \$15.87, 8% is \$7.44, 10% is approximately zero. At 6% the first investment is preferred while at 8% the second is preferred, and the difference is because the first project has the receipts concentrated in the later years which will be worth little at higher interest rates. The IRR is 10% for both projects.

6. \$20.79

$$7. P = \frac{3(1-1.07^{-10})}{.07} + 100(1.07)^{-10} = \$71.91$$

$$8. P = \frac{3(1-1.07^{-5})}{.07} + 100(1.07)^{-5} = \$83.59$$

$$P = \frac{3(1-1.10^{-5})}{.10} + 100(1.10)^{-5} = \$73.46$$

$$P = \frac{3(1-1.07^{-5})}{.07} + 73.46(1.07)^{-5} = \$64.68$$

9. Cost = \$500, Rate =  $33\frac{1}{2}\%$  straight line

Year	\$ Depreciation charge	\$ Accumulated depreciation	\$ Book value at end of year
1	133	133	367
2	133	267	233
3	133	400	100

10. Cost = \$500, Rate = 41% diminishing value

	\$	\$	\$
Year	Depreciation charge	Accumulated depreciation	Book value at end of year
1	205	205	295
2	121	326	174
3	71	397	103

$$r = 1 - \left( \frac{100}{500} \right)^{1/3} = 41\%$$

11. Annual repayment =  $\frac{150 \times .07}{1 - 1.07^{-3}} = \$57.16$

	\$				\$
	Loan outstanding at start of year	\$	\$	\$	Loan outstanding at end of year
Year		Repayment	Interest	Principal	
1	150.00	57.16	10.50	46.66	103.34
2	103.34	57.16	7.23	49.93	53.41
3	53.41	57.16	3.75	53.41	0

12. New annual repayment =  $\frac{103.34 \times .09}{1 - 1.09^{-2}} = \$58.75$

The amount will be an annuity of the difference in annual repayments.

$$\text{This is} = \frac{1.59[1 - (1.09)^{-2}]}{.09} = \$2.80$$

## CHAPTER 8

- Mode = 11, median = 12, mean = 12.64, variance = 25.00, standard deviation = 5.01, covariance = 0.927, correlation coefficient = 0.963.
- The regression equation is  $y_i = a + bx_i + \epsilon_i$ .  
OLS estimates are  $a = 2.27$ ,  $b = 2.07$  with standard errors of 1.06 and .1938. The  $t$  statistic for  $b$  is 10.7 and  $R^2$  is .927.